Simplified replica treatment of various random-energy and random-field models with confinement potential

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A modified version of the replica method recently proposed by the authors [J. Phys. A **32**, 585 (1999)] is applied to a certain class of models describing the motion of a particle in a random potential with superposed parabolic confinement. Different types of randomness are considered. Previous treatments of analogous models were based on replica symmetry breaking (RSB). The aim of the paper is to demonstrate that the new method provides reliable results for the average free energy and related quantities without taking into account RSB. The comparison with computer simulations shows satisfactory agreement. [S1063-651X(99)00810-7]

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I. INTRODUCTION

In a previous paper [1] we presented a new version of the replica method for the treatment of random systems. The basic idea of this version is to determine the distribution function of the free energy F directly from moments of the partition function $\langle Z^n\rangle$. It was shown that this method is well suited to circumvent the complicated problem of replica symmetry breaking (RSB). In the case of the random-energy model (REM) exact results could be derived from replica symmetric approximations of the moments $\langle Z^n\rangle$. For the Sherrington-Kirkpatrick (SK) model we found reasonable approximations without taking into account RSB. The aim of the present paper is to apply the modified method to a class of models which we call ''random-energy'' and ''random-field models with confinement'' and which were investigated previously by Mézard and Parisi [2], Engel [3], and Bouchaud and Mézard [4]. The models can be interpreted in terms of a particle moving one dimensionally in a potential which is given as the superposition of a parabolic confinement potential and a random part. Different assumptions about the statistics of the random potential lead to several variants of the problem.

Whereas the original REM was introduced by Derrida [5] as a model for spin glasses, the extensions considered in this paper are rather motivated by the problem of manifolds, interfaces, or fibers in random media [2,6,7]. The moving particle can be considered as the zero-dimensional limit of a manifold and the parabolic confinement potential stands for the surface or line energy of extended objects.

In the papers [2,3] Parisi's scheme of RSB was used to estimate different quantities for these models. Bouchaud and Mézard [4] investigated various random energy models with confinement by means of extreme value statistics and one-step RSB. We shall demonstrate that our method yields very satisfactory estimates on the basis of rather simple replica symmetric approximations of the moments $\langle Z^n\rangle$.

In Sec. II the fundamental equations developed in [1] are shortly summarized. In Sec. III the random-energy model with confinement and Gaussian distribution of the energies is considered. The extension to more general distribution functions is discussed in Sec. IV and results previously obtained in [4] are reproduced. Finally, Sec. V deals with a random-field model which is analogous to models already investigated in [2,3] and previously in [7].

II. BASIC EQUATIONS

The principal problem is to calculate the mean free energy of a disordered system, $\langle F\rangle = -T\langle \ln Z\rangle$, where Z is the partition function and the brackets denote the disorder average. The starting quantities are the moments $\langle Z^n\rangle =: \exp\Phi(n)$ which are assumed to be known in a sufficiently good approximation. In the conventional replica trick the mean free energy is obtained from a continuation of $\Phi(n)$ towards the limit $n=0$,

$$\langle \ln Z\rangle = \lim_{n\to 0}\frac{\langle Z^n\rangle - 1}{n} = \left.\frac{d\Phi}{dn}\right|_{n=0}. \tag{1}$$

However, in many interesting cases the function $\Phi(n)$ undergoes ''phase transitions'' in the interval $0<n<1$. Then a naive continuation from integer values of n towards $n=0$ leads to erroneous results. The phase transitions are connected with the phenomenon of RSB. Parisi [8,9] invented a famous scheme to take into account RSB in the continuation of $\Phi(n)$.

Alternatively, we try to reconstruct the distribution function of $Y=\ln Z=-\beta F$ ($\beta=1/T$) from the moments of Z. If we denote this distribution function by $w(Y)=\exp[-R(Y)]$, then the relation between the functions $\Phi(n)$ and $R(Y)$ is mainly given by a Legendre transformation with additional finite-size corrections [1]

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$$\frac{dR}{dY}=n\rightarrow Y(n)=\Phi'-\frac{1}{2}\frac{Y''}{Y'}\approx\Phi'-\frac{1}{2}\frac{\Phi'''}{\Phi''},$$
$$R(Y(n))=R(n)=n\Phi'-\Phi+\frac{1}{2}\ln 2\pi\Phi''-\frac{n}{2}\frac{\Phi'''}{\Phi''}, \tag{2}$$

where the primes denote derivatives with respect to n. In reasonable models R and Φ are extensive quantities and the last terms in Eqs. (2) as well as the logarithmic contribution are indeed finite-size corrections. The normalization condition for the distribution function $w(Y)$ leads to a lower cutoff at $Y(n_0)$ where n_0 is given by

$$R(n_0)+\ln n_0=0. \tag{3}$$

Due to the cutoff at finite n_0, the limit $n\rightarrow 0$ in the function $\Phi(n)$ need not be performed and therefore RSB becomes less important. This is a main advantage of the proposed procedure in comparison with the usual one described by Eq. (1). The cutoff of the distribution function $w(Y)$ and the RSB transition in $\Phi(n)$ even seem to correspond to each other to some extent.

Disregarding finite-size corrections, we may identify the mean free energy with the cutoff value

$$-\langle F\rangle=\frac{1}{\beta}\langle Y\rangle\approx\frac{1}{\beta}Y(n_0)=:\frac{1}{\beta}Y_0. \tag{4}$$

III. RANDOM-ENERGY MODEL WITH CONFINEMENT

In [1] the above-described procedure has been shown to yield very good results for the REM. This model is defined by a partition function [5]

$$Z=\sum_{k=1}^{2^N} e^{-\beta E_k}, \tag{5}$$

where the E_k are independent random variables with Gaussian distribution. Instead of Eq. (5) we shall now consider a model with an infinite number of states k. The number of accessible states, however, is limited by an additional parabolic confinement potential

$$Z=\sum_{k=-\infty}^{\infty} e^{-\beta[E_k+(\mu/2)k^2]}. \tag{6}$$

The energies E_k are again independent random variables with Gaussian distribution, vanishing mean value $\langle E_k\rangle=0$, and variance $\langle E_k^2\rangle=1$. The parameter μ controls the effective number of states really involved. Small values of μ correspond to large systems. The model (6) is a discrete version of the continuous models studied in [3,4].

The calculation can be carried out along the same lines as for the basic REM in [1]. At very low temperatures $\beta\rightarrow\infty$ the partition function Z is dominated by a single ground state term with smallest energy. Then the n-fold sum in

$$\langle Z^n\rangle=\left\langle \sum_{k_1,\ldots,k_n}\exp\left[-\beta(E_{k_1}+\cdots+E_{k_n})-\frac{\mu\beta}{2}(k_1^2+\cdots+k_n^2)\right]\right\rangle \tag{7}$$

reduces approximately to a simple sum over $k_1=k_2=\cdots=k_n=k$:

$$\langle Z^n\rangle\approx\left\langle\sum_k e^{-n\beta[E_k+(\mu/2)k^2]}\right\rangle=e^{(n\beta)^2/2}\sum_k e^{-(n\beta\mu/2)k^2}$$
$$\approx e^{(n\beta)^2/2}\int_{-\infty}^{\infty}dk e^{-(n\beta\mu/2)k^2}=\sqrt{\frac{2\pi}{n\beta\mu}}e^{(n\beta)^2/2},$$
$$\Phi(n)=\frac{n^2\beta^2}{2}-\frac{1}{2}\ln\frac{n\beta\mu}{2\pi}. \tag{8}$$

The transition to the integral over k is only justified for sufficiently small $n\beta\mu<1$.

The simple replica trick (1) fails for expression (8) because of the divergence of $\Phi(n)$ at $n=0$. The transformation to $Y(n)$ and $R(n)$ yields

$$Y(n)=n\beta^2-\frac{1}{2n}+\frac{1}{n(2n^2\beta^2+1)},$$
$$R(n)=\frac{1}{2}\left[n^2\beta^2-1+\ln n\beta^3\mu\left(1+\frac{1}{2n^2\beta^2}\right)\right]+\frac{1}{2n^2\beta^2+1}. \tag{9}$$

Equation (3) for the cutoff value $n_0=\gamma/\beta$ takes the form

$$\gamma^2+\ln\gamma^3\mu+\frac{2}{2\gamma^2+1}-1+\ln\left(1+\frac{1}{2\gamma^2}\right)=0. \tag{10}$$

According to Eq. (4), the mean free energy is then given by

$$-\langle F\rangle=\frac{1}{\beta}Y(n_0)=\gamma-\frac{1}{2\gamma}+\frac{1}{\gamma(2\gamma^2+1)}. \tag{11}$$

This result does not depend on the temperature. The system is frozen. Therefore the entropy vanishes, $\langle S\rangle=0$, and the energy coincides with the free energy, $\langle U\rangle=\langle F\rangle$. In the limit of a very broad confinement potential $\mu\rightarrow 0$ the solution behaves as

$$\gamma\rightarrow\sqrt{|\ln\mu|},$$
$$-\langle F\rangle=-\langle U\rangle\approx\gamma\rightarrow\sqrt{|\ln\mu|}. \tag{12}$$

For high temperatures the situation becomes very simple. The replicas are nearly independent from each other and the approximation

$$\langle Z^n\rangle\approx\langle Z\rangle^n,$$
$$\Phi(n)=n\Phi(1)=\frac{n}{2}\left(\beta^2-\ln\frac{\beta\mu}{2\pi}\right) \tag{13}$$

can be used. Then the original replica trick (1) can be applied in a naive way. RSB does not occur. In this simple situation

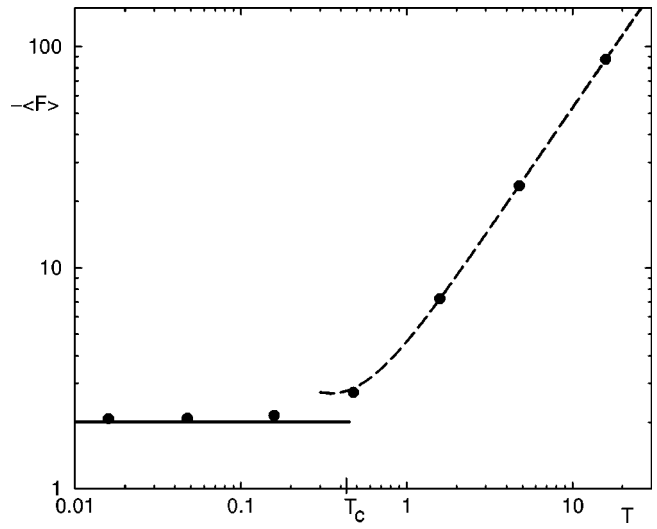

FIG. 1. Mean free energy of the REM with confinement (6) for $\mu = 0.001\,58$. Solid line: analytical result for low temperatures (11); dashed line: high-temperature approximation (14); dots: computer simulations averaged over 600 samples.

the present method leads to the same expessions as usual procedures. In the approximation (13) the function

$$Y(n) = \frac{1}{2}\left(\beta^2 - \ln\frac{\beta\mu}{2\pi}\right)$$

becomes independent of n which means that the distribution function of Y reduces to a δ-like function. Therefore the mean value is simply

$$-\langle F\rangle = \frac{1}{\beta}\langle Y\rangle = \frac{1}{2}\left(\beta - \frac{1}{\beta}\ln\frac{\beta\mu}{2\pi}\right). \tag{14}$$

Entropy and energy are given by

$$\langle S\rangle = \beta^2\frac{\partial\langle F\rangle}{\partial\beta} = \frac{1}{2}\left(1 - \beta^2 - \ln\frac{\beta\mu}{2\pi}\right),$$
$$\langle U\rangle = \langle F\rangle + \beta\frac{\partial\langle F\rangle}{\partial\beta} = \frac{1}{2\beta} - \beta. \tag{15}$$

The transition between the low- and high-temperature behavior occurs in the vicinity of $\beta_c = \gamma$ or $n_0 = 1$. Of course, due to the effective finiteness of the system, the transition is not sharp. In Fig. 1 the analytical estimates (11) and (14) are compared to computer simulations for $\mu = (2\pi)^{1/4}\times 10^{-3} = 0.001\,58$ and $|k| \leqslant 200$, averaged over 600 samples. Statistical errors are smaller than the size of the dots. This is also true for the following figures. The agreement for $T \to 0$ and $T \gg T_c$ is quite good. Deviations occur in the transition region $T \approx T_c = 1/\gamma$ where the high- and the low-temperature approximations for $\Phi(n)$ are not appropriate.

A further interesting quantity is the mean square displacement of the system in the confinement potential due to the random energies and thermal motion. It can be calculated immediately from the free energy. For a given realization of the system one finds

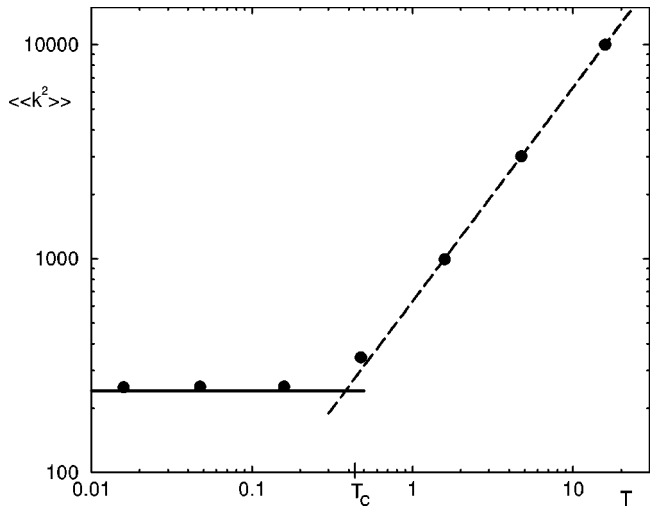

FIG. 2. Mean square displacement of the REM with confinement (6) for $\mu = 0.001\,58$. Solid line: analytical result for low temperatures (18); dashed line: high-temperature approximation (20); dots: computer simulations averaged over 600 samples.

$$\langle k^2\rangle = \frac{1}{Z}\sum_k k^2 e^{-\beta[E_k + (\mu/2)k^2]} = -\frac{2}{\beta}\frac{\partial}{\partial\mu}\ln Z = 2\frac{\partial F}{\partial\mu}, \tag{16}$$

where the brackets denote the thermodynamic average over the canonical ensemble. A subsequent disorder average leads to

$$\langle\langle k^2\rangle\rangle = 2\frac{\partial\langle F\rangle}{\partial\mu}. \tag{17}$$

Application of this formula to the low-temperature result (11) gives

$$\langle\langle k^2\rangle\rangle = 2\frac{d\gamma}{d\mu}\frac{d\langle F\rangle}{d\gamma} = \frac{1}{\gamma\mu}\frac{8\gamma^6 + 12\gamma^4 - 6\gamma^2 - 1}{8\gamma^6 + 20\gamma^4 + 2\gamma^2 + 1}$$
$$\approx \frac{1}{\gamma\mu}\left[1 - \frac{1}{\gamma^2} + O\left(\frac{1}{\gamma^4}\right)\right]. \tag{18}$$

In the limit of small μ this expression tends to

$$\langle\langle k^2\rangle\rangle \to \frac{1}{\mu\sqrt{|\ln\mu|}} \quad \text{for} \quad \mu \to 0. \tag{19}$$

The asymptotic behavior described by Eqs. (12) and (19) agrees with the estimates obtained previously in [3].

For the high-temperature regime one obtains

$$\langle\langle k^2\rangle\rangle = \frac{1}{\beta\mu}. \tag{20}$$

This corresponds to the Boltzmann distribution in the confinement potential. The random energies E_k lose their influence at high temperatures. In Fig. 2 the analytical results are compared to the same numerical simulations as in Fig. 1. Again, one observes excellent agreement in the low- and high-temperature limits.

IV. NON-GAUSSIAN DISTRIBUTIONS

In [4] Bouchaud and Mézard studied the model (6) for random energies E_k obeying a non-Gaussian statistics asymptotically described by a distribution function

$$P(E)\sim\exp(-B|E|^{\delta}) \quad \text{for} \quad E\to-\infty. \tag{21}$$

Moments $\langle Z^n\rangle$ of the partition function do only exist for $\delta>1$. In the following we shall extend the estimations performed in the preceding section to this more general case. The calculation of the moments $\langle Z^n\rangle$ in analogy to Eq. (8) leads to expressions of the type

$$\begin{aligned}\langle e^{-n\beta E}\rangle &\sim \int dx e^{n\beta x}e^{-Bx^{\delta}}\\ &\sim[B\delta(\delta-1)\gamma^{2-D}]^{-1/2}e^{(\delta-1)B\gamma^D},\end{aligned} \tag{22}$$

where the integral is performed by the method of steepest descent. The parameters γ and D are defined by

$$\gamma=\frac{n\beta}{B\delta}, \quad D=\frac{\delta}{\delta-1} \tag{23}$$

and γ^D characterizes the energy which yields a maximal contribution to the integral. The Gaussian case is recovered for

$$\delta=2,\ B=\frac{1}{2},\ \gamma=n\beta,\ \langle e^{-n\beta E}\rangle=e^{n^2\beta^2/2}. \tag{24}$$

Thus, the generalization of Eq. (8) is given by the replacement

$$\begin{aligned}\frac{n^2\beta^2}{2}&\to(\delta-1)B\gamma^D+\frac{D-2}{2}\ln\gamma-\frac{1}{2}\ln B\delta(\delta-1),\\ \Phi(n)&=(\delta-1)B\gamma^D+\frac{D-2}{2}\ln\gamma-\frac{1}{2}\ln\frac{\mu n\beta B\delta(\delta-1)}{2\pi}+c,\end{aligned} \tag{25}$$

where c is a constant of order unity. Transformation (2) leads to

$$\begin{aligned}Y(n)&=\beta\gamma^{D-1}-\frac{1}{2n}+O\left(\frac{\gamma^{-D}}{Bn}\right),\\ R(n)&=B\gamma^D+\frac{1}{2}\ln\gamma+\frac{1}{2}\ln\mu\beta^2B\delta+O(1).\end{aligned} \tag{26}$$

The cutoff is determined by Eq. (3), which now reads

$$\begin{aligned}\gamma_0&=\frac{n_0\beta}{B\delta},\\ 0&=B\gamma_0^D+\frac{1}{2}\ln\mu+\frac{3}{2}\ln(\gamma_0B\delta)+O(1).\end{aligned} \tag{27}$$

For the free energy we obtain

$$-\langle F\rangle\approx\frac{Y(n_0)}{\beta}=\gamma_0^{D-1}-\frac{1}{2\gamma_0B\delta}+O\left(\frac{1}{\gamma_0^{D+1}(B\delta)^2}\right). \tag{28}$$

This low-temperature approximation does not depend on the temperature and describes a frozen state. In the limit of a large system with very broad confinement potential $\mu\to0$ the solution of Eq. (27) tends to

$$\gamma_0^D\to E_c^{\delta}-\frac{3}{2B}\ln\beta_c, \tag{29}$$

where the abbreviations

$$\frac{1}{2B}|\ln\mu|=:E_c^{\delta}, \quad B\delta E_c^{\delta-1}=:\beta_c \tag{30}$$

are used. Then we may write for the average energy

$$-\langle F\rangle=E_c-\frac{3}{2\beta_c}[\ln\beta_c+O(1)]. \tag{31}$$

The expression agrees with the result for the ground state energy given in Eq. (43) of [4] apart from the prefactor (-3) in the logarithmic term of Eq. (31) which seems to be missing in [4] by error.

The high-temperature approximation is easily obtained by

$$\begin{aligned}\langle Z^n\rangle&\approx\langle Z\rangle^n, \quad \Phi(n)=n\Phi(1),\\ Y(n)&=\frac{d\Phi}{dn}=\Phi(1)=\langle Y\rangle,\\ -\langle F\rangle&=\frac{\langle Y\rangle}{\beta}\approx\frac{\delta-1}{\delta}\left(\frac{\beta}{B\delta}\right)^{1/(\delta-1)}\\ &\quad-\frac{1}{2\beta}\ln\left[\frac{\mu\beta^2}{2\pi}\left(\frac{B\delta}{\beta}\right)^{1/(\delta-1)}\right].\end{aligned} \tag{32}$$

The transition between the frozen state and the high-temperature regime occurs at $\beta\approx\beta_c$ with β_c defined in Eq. (30). The spatial distribution of the occupied states in the confinement potential is characterized by $\langle\langle k^2\rangle\rangle$ and can be calculated according to Eq. (17). For the low-temperature case we find

$$\langle\langle k^2\rangle\rangle=2\frac{d\gamma_0}{d\mu}\frac{d\langle F\rangle}{d\gamma_0}=\frac{1}{\mu\beta_c}\left[1+\frac{3}{2}\frac{\delta-1}{\beta_cE_c}\ln\beta_c+O\left(\frac{1}{\beta_cE_c}\right)\right]. \tag{33}$$

The leading term coincides with the result given in Eq. (41) of [4]. In the high-temperature limit we obtain

$$\langle\langle k^2\rangle\rangle=2\frac{\partial\langle F\rangle}{\partial\mu}=\frac{1}{\beta\mu} \tag{34}$$

in agreement with the previous result (20) for the Gaussian case.

V. RANDOM-FIELD MODEL

In the preceding sections the energies E_k were considered as independent random variables. Let us now move to correlated energies. Instead of the energies E_k themselves the changes of the energy between neighboring states $E_k-E_{k-1}=\eta_k$ are chosen as independent random variables

with Gaussian distribution. Thus instead of a ''random potential'' we consider ''random fields'' with the following properties:

$$E_k=\sum_{k'=1}^{k}\eta_{k'},\ \langle\eta_k\rangle=0,\ \langle\eta_k\eta_{k'}\rangle=\delta_{kk'},$$
$$\langle E_kE_{k'}\rangle=k\quad(k'\geqslant k\geqslant 0). \tag{35}$$

A continuous model of this kind has been investigated already in [2,3] within the RSB scheme. The estimation of the moments $\langle Z^n\rangle$ is a little bit more involved than in Sec. III. For simplicity we restrict the partition function to positive values of k,

$$Z=\frac{1}{2}+\sum_{k=1}^{\infty}e^{-\beta[E_k+(\mu/2)k^2]}=:{\sum_k}'\,e^{-\beta[E_k+(\mu/2)k^2]},$$
$$\langle Z^n\rangle={\sum_{k_1\cdots k_n}}'\left\langle e^{-\beta\sum_{m=1}^{n}E_{k_m}}\right\rangle\exp\left[-\frac{\beta\mu}{2}\sum_{m=1}^{n}k_m^2\right]. \tag{36}$$

For arguments arranged according to $k_1\leqslant k_2\leqslant\cdots\leqslant k_n$ one finds

$$\sum E_{k_m}=n\sum_{k=1}^{k_1}\eta_k+(n-1)\sum_{k=k_1+1}^{k_2}\eta_k+\cdots+\sum_{k=k_{n-1}+1}^{k_n}\eta_k,$$
$$\left\langle e^{-\beta\Sigma E_{k_m}}\right\rangle=\exp\left(\frac{\beta^2}{2}[n^2k_1+(n-1)^2(k_2-k_1)+\cdots+(k_n-k_{n-1})]\right). \tag{37}$$

For parameter values $\beta<1$, $\beta\mu\ll 1$ the sums in Eq. (36) can be approximately replaced by integrals

$$\langle Z^n\rangle\approx n!\int_0^{\infty}dk_1\int_{k_1}^{\infty}dk_2\cdots\int_{k_{n-1}}^{\infty}dk_n\exp\left[\frac{\beta^2}{2}n^2k_1+\frac{\beta^2}{2}\sum_{m=1}^{n-1}(n-m)^2(k_{m+1}-k_m)-\frac{\beta\mu}{2}\sum_{m=1}^{n}k_m^2\right]. \tag{38}$$

The ordering of the arguments is compensated by the prefactor $n!$. For not too high temperatures $T=1/\beta$ the main contributions to the integrals in Eq. (38) result from the region $k_1\approx k_2\approx\cdots\approx k_n\approx k_0=\beta n/2\mu$ for which the integrand takes maximal values. Now starting the integrations with $k_n=k_{n-1}+\kappa$ and using the approximation

$$\int_0^{\infty}d\kappa e^{-\gamma\kappa-\delta\kappa^2}\approx\frac{1}{\gamma}\exp\left[-\frac{2\delta}{\gamma^2}\right], \tag{39}$$

which is valid for $\delta\ll\gamma^2$, we find

$$\int_{k_{n-1}}^{\infty}dk_n\exp\frac{\beta}{2}\left[\beta(k_n-k_{n-1})-\frac{\beta\mu}{2}k_n^2\right]\approx\frac{1}{\beta(\mu k_{n-1}-\beta/2)}\exp\left[-\frac{\beta\mu}{2}k_{n-1}^2-\frac{\mu}{\beta(\mu k_{n-1}-\beta/2)^2}\right]. \tag{40}$$

If we replace the variable k_{n-1} in the slowly varying quantities by $k_0=\beta n/2\mu$, expression (40) simplifies to

$$\frac{2}{\beta^2(n-1)}\exp\left[-\frac{\beta\mu}{2}k_{n-1}^2-\frac{4\mu}{\beta^3(n-1)^2}\right]. \tag{41}$$

Performing the following integrations in the same manner we finally end up with the last integral over $k_1=k$:

$$\langle Z^n\rangle=n!\left(\frac{2}{\beta^2}\right)^{n-1}\prod_{m=1}^{n-1}\frac{1}{m(n-m)}\exp\left(-\frac{4\mu}{\beta^3}A(n)\right)\times\int_0^{\infty}dke^{(\beta/2)[\beta n^2k-\mu nk^2]}$$
$$\approx\left(\frac{2}{\beta^2}\right)^{n-1}\frac{n}{(n-1)!}\sqrt{\frac{2\pi}{\mu n\beta}}\exp\left[\frac{(n\beta)^3}{8\mu}-\frac{4\mu}{\beta^3}A(n)\right],\qquad A(n)=\sum_{m=1}^{n-1}\frac{1}{m(n-m)^2}=\frac{1}{n}\sum_{m=1}^{n-1}\left(\frac{2}{nm}+\frac{1}{m^2}\right). \tag{42}$$

In order to continue this expression to noninteger values $n\ll 1$, we use the representations

$$(n-1)!=\Gamma(n)\to\frac{1}{n}\quad(n\ll 1),$$
$$\frac{1}{m}=\int_0^{\infty}dxe^{-mx},\ \frac{1}{m^2}=\int_0^{\infty}dxxe^{-mx},$$
$$A(n)=-\frac{1}{n}\int_0^{\infty}dx\left[\frac{2}{n}+x\right]\frac{e^{-nx}-e^{-x}}{1-e^{-x}}\to-\frac{3}{n^3}\quad(n\ll 1). \tag{43}$$

Thus, the approximation for the moments $\langle Z^n\rangle$ in the region $n\ll 1$ finally reads

$$\Phi(n)=\ln\langle Z^n\rangle=\frac{(n\beta)^3}{8\mu}+\frac{1}{2}\ln\frac{2\pi}{\mu n\beta}+(n-1)\ln\frac{2}{\beta^2}+\ln n^2+\frac{12\mu}{(n\beta)^3}. \tag{44}$$

Now the formalism given in Sec. II can be applied. With the abbreviations

$$\frac{(n\beta)^3}{2\mu}=:\nu^3,\quad\varphi(\nu):=\frac{\nu^3}{4}+\frac{1}{2}\ln\pi\nu^3+\frac{6}{\nu^3} \tag{45}$$

one obtains

$$\Phi = \varphi(\nu) - n \ln \frac{\beta^2}{2},$$
$$Y = \frac{d\Phi}{dn} = \frac{\nu}{n}\left(\varphi' - \frac{1}{2}\frac{\varphi'''}{\varphi''} \right) - \ln \frac{\beta^2}{2}, \tag{46}$$
$$R + \ln n = \nu\left(\varphi' - \frac{1}{2}\frac{\varphi'''}{\varphi''} \right) - \varphi(\nu) + \frac{1}{2}\ln 2\pi\nu^2\varphi'' = 0.$$

The last equality represents the cutoff condition. It yields $\nu = n_0\beta/(2\mu)^{1/3} = 1.540$. Inserting this result in the foregoing line of Eq. (46) gives the mean free energy

$$-\langle F \rangle = \frac{Y_0}{\beta} = \frac{C}{\mu^{1/3}} - \frac{1}{\beta}\ln \frac{\beta^2}{2}, \tag{47}$$

with the constant

$$C = \frac{1}{2^{1/3}}\left(\varphi' - \frac{1}{2}\frac{\varphi'''}{\varphi''} \right) = 0.657. \tag{48}$$

The mean square displacement is determined by

$$\langle\langle k^2 \rangle\rangle = 2\frac{\partial \langle F \rangle}{\partial \mu} = \frac{2}{3}\frac{C}{\mu^{4/3}}. \tag{49}$$

At low temperatures $T = 1/\beta < 1$ the system is frozen in a state of minimal energy whose average value is $C/\mu^{1/3}$. For finite temperatures there is a thermal excitation around this ground state described by the last term in Eq. (47). This term is independent of μ because small excitations do not feel the slowly varying confinement potential. For the same reason the excitations are nearly symmetric with respect to the ground state and do not change the average displacement (49). The scaling of the ground state properties with the confinement parameter μ described by Eqs. (47) and (49) is the same as obtained in [2,3] by scaling arguments and by a RSB treatment and as it was discussed previously by Villain [6] and Villain *et al.* [7].

At very low temperatures $1/\beta \ll 1$ the behavior of the free energy (47) cannot be completely correct since it leads to negative values of the entropy

$$S = -\frac{\partial \langle F \rangle}{\partial T} = \frac{\partial}{\partial T} T \ln 2T^2 = 2 + \ln 2T^2. \tag{50}$$

The reason is that the replacement of the sums by integrals in the transition from Eq. (37) to Eq. (38) breaks down in the region $\beta > 1$ (dotted curve in Fig. 3).

The analytical estimates (47) and (49) are compared to the results of computer simulations in Figs. 3 and 4, respectively. The confinement potential is chosen as $\mu = 10^{-3}$ and k values up to 300 are taken into account. The results are averaged over 5000 samples. Again, statistical errors are smaller than the size of the dots in the figures. The agreement is very satisfactory in the low-temperature region. Deviations occur for higher temperatures. Above a characteristic temperature $T_c \approx \nu^{-1}(2\mu)^{-1/3}$ (given in a similar form in [2,3]) a transition to a high-temperature regime takes place. For $T \to T_c$ the cutoff parameter n_0 tends to unity. The examples discussed in [1] already showed that, in the thermodynamic limit, the phase transition to the high-temperature phase occurs exactly at $n_0 = 1$. In the present treatment the breakdown of the approximations for $n_0 \to 1$ can clearly be seen in Eq. (43).

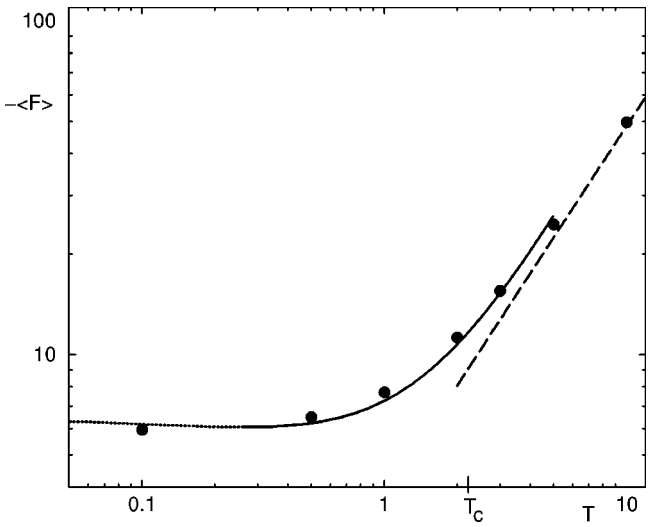

FIG. 3. Mean free energy of the random-field model for $\mu = 0.001$. Solid line: analytical result for low temperatures (47); dashed line: high-temperature approximation (51); dots: computer simulations averaged over 5000 samples. The dotted line indicates the failure of the approximation for very low temperatures.

The high-temperature behavior for $T > T_c$ is dominated by the confinement potential. In the crudest approximation the random energies can be entirely omitted. Then no replica treatment is needed. The partition function (36) and the free energy reduce to

$$\langle Z \rangle = Z = \frac{1}{2} + \sum_{k=1}^{\infty} e^{-(\beta\mu/2)k^2} \approx \int_0^{\infty} dk e^{-(\beta\mu/2)k^2} = \sqrt{\frac{\pi}{2\beta\mu}},$$
$$\langle F \rangle = F = -T \ln Z = -\frac{T}{2}\ln \frac{\pi T}{2\mu} \tag{51}$$

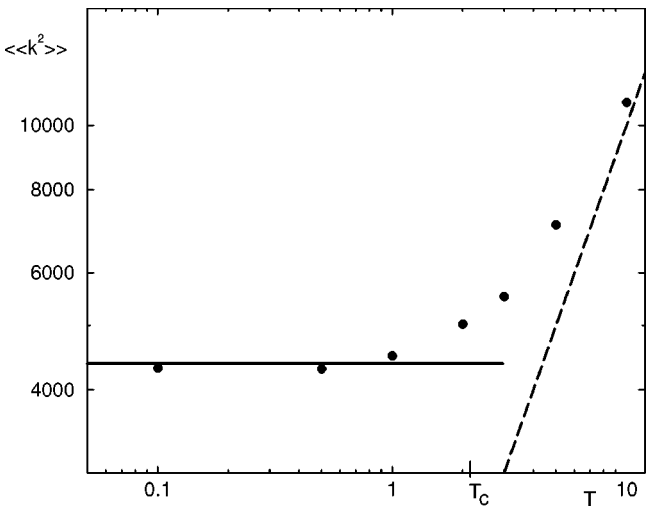

FIG. 4. Mean square displacement of the random-field model for $\mu = 0.001$. Solid line: analytical result for low temperatures (49); dashed line: high-temperature approximation (52); dots: computer simulations averaged over 5000 samples.

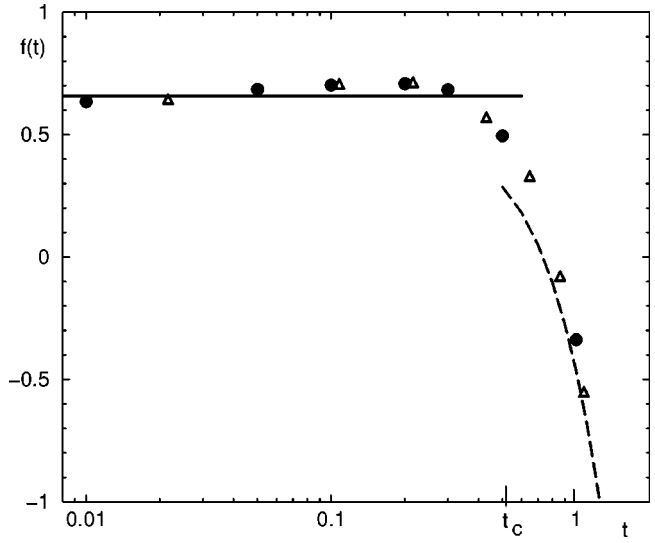

FIG. 5. Energy parameter $f(t)$ versus rescaled temperature t. Solid line: analytical result for low temperatures (48); dashed line: high-temperature approximation (54); dots: computer simulations for $\mu=0.001$ averaged over 5000 samples; triangles: computer simulations for $\mu=0.01$ averaged over 6000 samples.

and the mean square displacement is

$$\langle\langle k^2\rangle\rangle = 2\frac{\partial F}{\partial \mu} = \frac{T}{\mu}. \tag{52}$$

These approximations are also represented in Figs. 3 and 4 by dashed lines. They fit the simulations perfectly for $T \gtrsim 10$.

Finally, in Fig. 5 the comparison between the simulation results and the analytical estimates is made in a different form. The quantity

$$f(t) = \mu^{1/3}\left[-\langle F\rangle + \frac{1}{\beta}\ln\frac{\beta^2}{2}\right] \tag{53}$$

is plotted as a function of the rescaled temperature $t = \mu^{1/3}T$. According to Eq. (47) it should be equal to the μ-independent constant C for low temperatures. Therefore, simulation results for a different value of $\mu=10^{-2}$ are included. For high temperatures Eq. (51) yields the asymptotic behavior

$$f(t) = \frac{t}{2}\ln\frac{\pi}{(2t)^3} \quad (t \gg t_c). \tag{54}$$

The transition occurs in the vicinity of $t_c = \mu^{1/3}T_c \approx 0.52$. Again, the agreement is satisfactory.

VI. CONCLUSION

We have shown that the models considered in this paper can be treated by means of the simplified replica method developed in [1]. The remaining differences between computer simulations and analytical estimates for intermediate temperatures are undoubtedly due to the rough approximations used for the moments $\Phi(n)$ and not to the method itself.

The simulations are performed for rather small systems which are far from the ''thermodynamic limit'' $\mu \to 0$, $\langle\langle k^2\rangle\rangle \to \infty$. Therefore, the finite-size corrections contained in Eqs. (2) and (3) considerably influence the analytical estimates. Equations (2)–(4) rest on saddle-point expansions in various integrals over the distribution function $w(Y)$ which become correct in the thermodynamic limit. The conditions of validity for these expansions are given in [1]. A check of these conditions for the above examples shows that the expansion parameters, although smaller than unity, are not very small. Thus, inclusion of higher order terms could slightly change the estimates.

We think that the simple cutoff procedure used above corresponds to a one-step RSB [3,4,9]. The cutoff parameter n_0 can be shown to coincide with the step position in the overlap spectrum $q(x)$ of the RSB treatment. Certainly the simplified method, at least in its present form, is not able to reveal more complicated features of the overlap spectra $q(x)$ of glassy systems. However, a possible advantage of the present method could be that it does not necessarily demand Gaussian mean field approximations in the calculation of the moments $\langle Z^n\rangle$ as is the case in Parisi's RSB scheme.

Finally, it seems to us that the success of the present treatment in the limiting case of zero-dimensional models justifies the hope that more involved models for higher-dimensional manifolds could be dealt with in a similar way.

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